Random walks on multiconnected manifolds and conformal field theory

1994 J. Phys. A: Math. Gen. 27 2289

(http://iopscience.iop.org/0305-4470/27/7/010)

J. Phys. A: Math. Gen. 27 (1994) 2289–2298. Printed in the UK

Random walks on multiconnected manifolds and conformal field theory

Sergei Nechaev†§ and Anatolii Vershik‡||

† L D Landau Institute for Theoretical Physics, 117940 Moscow, Russia
and Lab. Physique Mathematique, Universite Montpellier II, Pl. E Bataillon Case 50, 34095 Montpellier Cedex 05, France
‡ St Petersburg branch of Steklov Mathematical Institute, Fontanka, 27, 191011 St Petersburg, Russia

Received 8 September 1993

Abstract. We propose a simple geometrical method which enables us to link the topological properties of a random walk on the double-punctured plane and the conformal field theory characterized by the central charge $c = -2$ and the conformal dimension $\Delta = -\frac{1}{8}$. We discuss briefly the connection between the topological invariants obtained from the conformal methods and the algebraic Alexander invariants for the simplest non-trivial braid B_3.

1. Introduction

In the last few years owing to the pioneering works of [1, 2] significant progress in understanding of the relation among Chern–Simons topological field theory, construction of algebraic knot and link invariants and the conformal field theory has been made. But despite the general concepts being well elaborated in the field–theoretic context, the application of these powerful ideas in related areas of mathematics and physics, such as, for instance, probability theory and polymer physics is highly restricted. In our opinion this is due to two facts: first, there is the problem of communication, i.e. the languages used by specialists in topological field theory and probability theory are completely different at first sight, and, second, there are no evident realizations of these field–theoretic ideas in simple geometrical examples for physical systems.

In the present paper we show, in the framework of the differential geometrical approach, the connection between the topological properties of a random walk on a double punctured plane and the behaviour of the four-point correlaton function in the conformal theory with central charge $c = -2$. Using the conformal methods we construct the non-Abelian topological invariants for entanglements of the random walk with the removed points on the plane; we also briefly discuss the relation of that problem with the random walk on the simplest non-trivial braid group B_3. A detailed paper devoted to the investigation of the topological properties of random walks on non-commutative groups [3] is now in progress.

§ E-mail address: rechaev@cpd.landau.free.net

|| E-mail address: vershik@lomi.spb.su

0305-4470/94/072289+10$19.50

2. Conformal mapping for a double-punctured plane

We consider the random walk of total length L with the effective elementary step of length l ($l \equiv 1$) *on the complex plane,* $\mathfrak{R}$, *with two points removed. Suppose the coordinates of* these points are $r_1 = (0,0)$ and $r_2 = (a,0)$ $(a \equiv 1)$. Such a choice does not mean a loss of generality because by means of simultaneous rescaling of the effective step, l, of the random walk and of the distance, a, between the obstacles we can always come to the case of any arbitrary values of l and a.

The topological constraints for the random walk mean that the loop on $\mathfrak{R}$ enclosing the points r_1 and/or r_2 cannot be continuously (without path rupture) contracted to the point.

In this paper we restrict ourselves to only the 'critical' case of infinitely long trajectories, i.e. we suppose $L \to \infty$. In field–theoretic language this means the consideration of the massless free field theory on $\mathfrak{R}$. Actually, the partition function for random walks on $\mathfrak{R}$ written in the second quantized form is generated by the scalar-field Hamiltonian $H = \frac{1}{2}(\nabla\varphi)^2 + m\varphi^2$ where the mass m is the chemical potential conjugated to the length of the path. Thus, for $L \to \infty$ we have $m_c = 0$ which just corresponds to the critical point in the conformal theories [4].

It was shown in the paper [5] that it is possible to construct the universal covering surface $\mathfrak{I}$ for $\mathfrak{R}$. We briefly describe the way of doing that. First, each point on $\mathfrak{R}$ and on $\mathfrak{I}$ we characterize by the complex coordinates $z = x + \mathrm{i}y$ and $\tau = u + \mathrm{i}v$ correspondingly. *Now we make three cuts on* $\mathfrak{R}$ *between* $(0,0)$ *and* $(0,1)$, *between* $(0,1)$ *and* (∞) *and* between (∞) and $(0,0)$ along the line $\operatorname{Im} z = 0$ as it is shown in figure 1(a). These cuts separate the upper ($\operatorname{Im} z > 0$) and lower ($\operatorname{Im} z < 0$) half-planes of the plane z. We perform the conformal transformation of the half-plane $\operatorname{Im} z > 0$ to the fundamental domain of $\mathfrak{I}$—the curvilinear zero-angled triangle lying in the half-plane $\operatorname{Im}\tau > 0$ of the plane τ—see figure 1(b). It has been shown in [6] that this transformation can be realized using modular function $k^2(\tau)$, i.e. we have

$$z(\tau) \equiv k^2(\tau) = \frac{\theta_2^4(0,\tau)}{\theta_3^4(0,\tau)} \tag{1}$$

where $\theta_2(0,\tau)$ and $\theta_3(0,\tau)$ are the elliptic Jacobi θ-functions. We recall their definitions

$$\theta_2(\zeta,\tau) = 2\mathrm{e}^{\mathrm{i}(\pi/4)\tau} \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}\pi\tau n(n+1)} \cos(2n+1)\zeta \qquad \theta_3(\zeta,\tau) = 1 + 2\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}\pi\tau n^2} \cos 2n\zeta\,.$$

It is not hard to proof that the universal covering $\mathfrak{I}$ is now realized as a whole upper half-plane $\operatorname{Im}\tau > 0$ and all images of peculiar (branching) points on $\mathfrak{R}$ are just transferred to the boundary $\operatorname{Im}\tau = 0$. Speaking more formally, each fundamental domain of $\mathfrak{I}$ should be considered as a Reeman sheet corresponding to the fibre bundle of $\mathfrak{R}$. The space $\mathfrak{I}$ has the discrete group of motions Γ_2 [7] generated by the basic substitutions

$$\tau' \to \tau + 2 \qquad \tau' \to \tau/(2\tau+1)\,. \tag{2}$$

3. Non-Abelian topological invariants for a random walk on $\mathfrak{R}$

We distinguish different topological states of the path with respect to the removed points on $\mathfrak{R}$ by the topological invariants constructed from the conformal mapping described above.

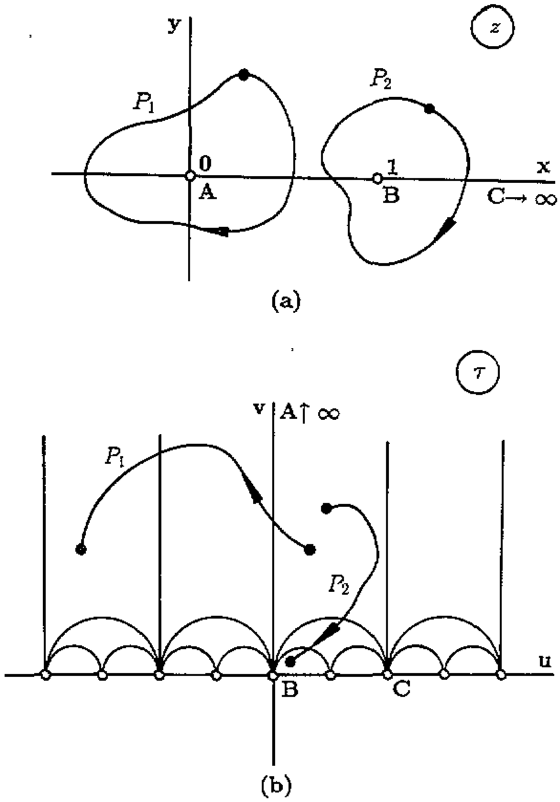

Figure 1. (*a*) Double-punctured complex plane, $\Re$, with two basic contours P_1 and P_2; (*b*) universal covering space, $\Im$, corresponding to the fibre bundle of $\Re$.

Shown in figure 2 are two closed contours C_1 and C_2 starting and ending at some arbitrary point $\boldsymbol{R}_0$ on $\Re$ which belongs to the same homology class but have different classes of homotopy. On the universal covering $\Im$ the coordinates of the initial and final points of the trajectory determine [5]: the corresponding Euclidean coordinates on $\Re$; the homotopy class of the path on $\Re$. In particular, the closed contours on $\Im$ correspond only to the *unentangled* closed contours on $\Re$.

The coordinates of the ends of the trajectory on $\Im$ we use as a topological invariant of the corresponding path on $\Re$. We should stress that this invariant is *complete* for our problem, it reflects the non-Abelian character of entanglements in the right way and could be used for the safe classification of homotopy class of the trajectories on $\Re$.

The function $z(\tau)$ has the inversed single-valued function $\tau(z) \equiv z^{-1}(\tau)$ defined in the basic domain of $\Im$—the triangle ABC. We consider the multivalued function $f(z)$ determined as follows:

- the function $f(z)$ coincides with $\tau(z)$ in the basic fundamental domain;
- in all other domains of the complex half-plane $\operatorname{Im} z > 0$ the function $f(z)$ is obtained from $\tau(z)$ by means of analytic continuation of $\tau(z)$ through the boundaries of these domains using the substitutions (2).

The explicit form of the function $f(z)$ reads

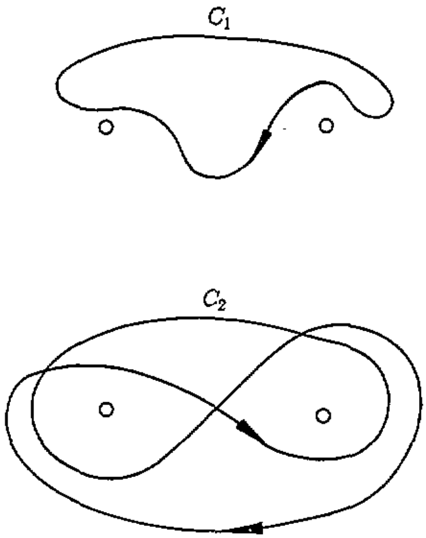

Figure 2. *Example of two contours belonging to the same class of homology but having different homotopy classes.*

$$f(z)=\frac{\displaystyle\int_1^{1/\sqrt{z}}\frac{\mathrm{d}\kappa}{\sqrt{(1-\kappa^2)(1-z\kappa^2)}}}{\displaystyle\int_0^1\frac{\mathrm{d}\kappa}{\sqrt{(1-\kappa^2)(1-z\kappa^2)}}}. \tag{3}$$

The topological invariant, $\mathrm{Inv}(C)$, for the closed contour, C, on the plane z we characterize by the values of the initial, $f_{\mathrm{in}}(z)$, and final, $f_{\mathrm{fin}}(z)$, points of the path on the half-plane $\mathrm{Im}\,\tau>0$ (see figure 1(b) and construct its complex realization, $\mathrm{Inv}_{(z)}(C)$, as a full derivative along the contour C:

$$\mathrm{Inv}_{(z)}(C)=f_{\mathrm{in}}-f_{\mathrm{fin}}=\int_C\frac{\mathrm{d}f(z)}{\mathrm{d}z}\,\mathrm{d}z\,. \tag{4}$$

In other words, the invariant, $\mathrm{Inv}(C)$, can be associated with the flux through the contour C on the plane (x,y):

$$\mathrm{Inv}(C)\equiv\mathrm{Inv}_{(x,y)}(C)=\int_C\nabla f(x,y)\boldsymbol{n}\,\mathrm{d}\boldsymbol{R} \tag{5}$$

where $\boldsymbol{n}$ is the unit vector normal to the curve C and $\mathrm{d}\boldsymbol{R}=e_x\,\mathrm{d}x+e_y\,\mathrm{d}y$ on the plane (x,y). Using the simple transformations

$$\boldsymbol{n}\,\mathrm{d}\boldsymbol{R}=e_x\,\mathrm{d}y-e_y\,\mathrm{d}x=\mathrm{d}\boldsymbol{R}\times\xi$$

and

$$\nabla f(x,y)(\mathrm{d}\boldsymbol{R}\times\xi)=(\xi\times\nabla f(x,y))\,\mathrm{d}\boldsymbol{R}$$

where $\xi = (0, 0, 1)$ is the unit vector normal to the plane (x, y), we can rewrite (5) in the following form:

$$\mathrm{Inv}(C) = \int_C \xi \times \nabla f(x, y) \boldsymbol{v} \, \mathrm{d}s \tag{6}$$

where $\boldsymbol{v} = \mathrm{d}\boldsymbol{R}/\mathrm{d}s$ is the 'velocity' along the trajectory and $\mathrm{d}s$ is the differential path length.

The vector product $\boldsymbol{A} = \boldsymbol{\xi} \times \nabla f(x, y)$ (where the function f is defined in (3)) we can consider as a non-Abelian generalization of the vector potential of the solenoidal 'magnetic fields' which are normal to the plane (x, y) and cross the plane in the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

4. Monodromy transformations and conformal field theory

Let us consider two basic contours P_1 and P_2 enclosing the branching points $z_1 = (0, 0)$ and $z_2 = (1, 0)$ on the plane z—see figure 1. The function $f(z)$ (equation (3)) obeys the following transformations:

$$f\left(z \xrightarrow{P_1} z\right) \to \tilde{f}_1(z) = \frac{a_1 f(z) + b_1}{c_1 f(z) + d_1} \tag{7a}$$

$$f\left(z \xrightarrow{P_2} z\right) \to \tilde{f}_2(z) = \frac{a_2 f(z) + b_2}{c_2 f(z) + d_2} \tag{7b}$$

where

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \hat{\sigma}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

are the matrices of basic substitutions of the group Γ_2 (they are consistent with the definition (2)).

We suppose $f(z)$ to be a ratio of two basic solutions, $u_1(z)$ and $u_2(z)$, of some second-order differential equation with peculiar points $\{z_1 = (0, 0), z_2 = (0, 1), z_3 = (\infty)\}$. From the analytic theory of differential equations [8] it is well known that the solutions $u_1(z)$ and $u_2(z)$ undergo the following linear transformations when the variable z moves along the contours P_1 and P_2:

$$P_1 : \begin{pmatrix} \tilde{u}_1(z) \\ \tilde{u}_2(z) \end{pmatrix} = \hat{\sigma}_1 \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} \qquad P_2 : \begin{pmatrix} \tilde{u}_1(z) \\ \tilde{u}_2(z) \end{pmatrix} = \hat{\sigma}_2 \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}. \tag{8}$$

The problem of restoring the form of the differential equation knowing the monodromy matrices $\hat{\sigma}_1$ and $\hat{\sigma}_2$ has an old history [8] and in our particular case has the solution

$$z(1-z)\frac{\mathrm{d}^2 u(z)}{\mathrm{d}z^2} + (1-2z)\frac{\mathrm{d}u(z)}{\mathrm{d}z} - \frac{1}{4}u(z) = 0. \tag{9}$$

Now let us ask the following question: is it possible to consider (9) as a degeneration equation determining the four-point correlation function of *some* conformal field theory? We show that the answer is positive and the coefficients in (9) completely define the corresponding Virasoro algebra.

We introduce the conformal operator, $\varphi(z)$, on the complex plane z. The dimension, Δ, of this operator we define from the conformal correlator

$$\langle\varphi(z)\varphi(z')\rangle \sim \frac{1}{|z-z'|^{2\Delta}}\,. \tag{10}$$

Let us suppose that $\varphi(z)$ is the primary field, then the four-point correlation function $\langle\varphi(z_1)\varphi(z_2)\varphi(z_3)\varphi(z_4)\rangle$ satisfies the degeneration equation following from the conformal Ward identity [4, 9, 10]. In the form of a Riemann ordinary differential equation this equation on the conformal correlator

$$\psi(z|z_1, z_2, z_3) = \langle\varphi(z)\varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle$$

with fixed points $\{z_1 = (0, 0), z_2 = (1, 0), z_3 = \infty\}$ reads [4, 9]

$$\left\{\frac{3}{2(2\Delta+1)}\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \frac{1}{z-1}\frac{d}{dz} - \frac{\Delta}{z^2} - \frac{\Delta}{(z-1)^2} + \frac{2\Delta}{z(z-1)}\right\}\psi(z|z_1, z_2, z_3) = 0\,.$$

Performing the substitution

$$\psi(z|z_1, z_2, z_3) = [z(z-1)]^{-2\Delta}\,u(z)$$

we get the equation

$$z(z-1)u''(z) - \tfrac{2}{3}(1-4\Delta)(1-2z)u'(z) + \tfrac{2}{3}(2\Delta - 8\Delta^2)u(z) = 0 \tag{11}$$

which coincides with (9) for one single value of Δ

$$\Delta = -\tfrac{1}{8}\,. \tag{12}$$

The conformal properties of the stress–energy tensor, $T(z)$, are defined by the coefficients, L_n, in its Lourant expansion, $T(z) = \sum_{n=-\infty}^{\infty} L_n/z^{n+2}$. These coefficients form the Virasoro algebra [4]

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}c(n^3 - n)\delta_{n+m,0}$$

where the parameter c is the central charge of the theory. Using the relation $c = 2\Delta(5-8\Delta)/(2\Delta+1)$ established in [9] and (12) we obtain

$$c = -2\,. \tag{13}$$

We find very interesting the fact mentioned by B Duplantier. He poined out that the value $\Delta = -\frac{1}{8}$ (equation (12)) coincides with the surface exponent (i.e. with the conformal *dimension of the two-point correlator near the surface) for the dense phase of the O($n = 0$)* lattice model describing the so-called 'Manhattan random walks' [11].

According to [11] the value $\Delta = -\frac{1}{8}$ belongs to the family of the critical exponents

$$x_S \equiv x^{\mathrm{surf}}_{O(n=0)} = \tfrac{1}{8}S(S-2) \tag{14}$$

where S is the number of fluctuating chains tightly together in the bunch connecting the points z and z' on the complex plane (see figure 3 and [11] for more details). The critical

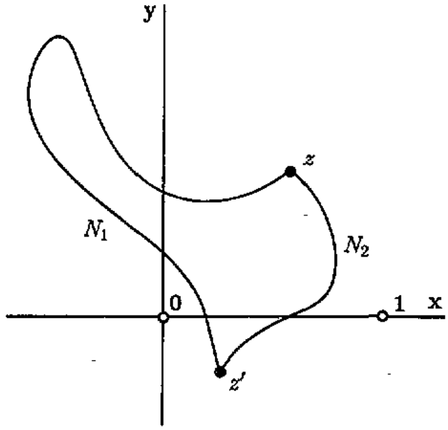

Figure 3. Simplest 'watermelon' configuration of the bunch of $S = 2$ chains having the trivial (i.e. contractable to the point) topological configuration with respect to the peculiar points on $\mathfrak{R}$.

behaviour of the two-point correlation function for the 'watermelon' configuration with S chains in the bunch has the following scaling form:

$$G(|z-z'|, m_c) \sim \frac{1}{|z-z'|^{2x_S}}. \tag{15}$$

The case $S = 1$ corresponds to the conformal dimension of the primary fields $\varphi(z)$ considered above (10)–(13). For $S = 2$ (14) gives $x_S = 0$ and hence we could expect the logarithmic behaviour of the correlation function (15). Using the results of the works [12] we can establish that fact directly for the contractible random loop on the plane with removed points.

The group Γ_2 has the metrics of the Cayley tree. Introducing the non-Euclidean distance on the Cayley tree, η, between the ends of the simplest watermelon configuration with $S = 2$ (see figure 3) we have [12]

$$G(\eta, N_1, N_2) \sim \frac{\eta^2}{N_1^{3/2} N_2^{3/2}} \exp\left\{-\frac{\eta^2}{2}\left(\frac{1}{N_1} + \frac{1}{N_2}\right)\right\} \tag{16}$$

where N_1 and N_2 are the lengths of the trajectories in the bunch. The mapping of the correlation function (16) of the contractible random walks on the universal covering $\mathfrak{I}$ onto the double-punctured complex plane $\mathfrak{R}$ is given by the convolution [12]

$$G(|z-z'|, N_1, N_2) \sim \int_0^\infty \frac{1}{\eta} \exp\left\{-\frac{|z-z'|^2}{\eta}\right\} G(\eta, N_1, N_2)\, \mathrm{d}\eta. \tag{17}$$

The critical behaviour of the correlation function (15) can be obtained using the Laplace transform:

$$G(|z-z'|, m_c) = \int G(|z-z'|, N_1, N_2)\, \mathrm{e}^{m_c(N_1+N_2)}\, \mathrm{d}N_1\, \mathrm{d}N_2. \tag{18}$$

Substituting (16), (17) into (18) we get

$$G(|z-z'|, m_c) \sim K_0\left(|z-z'|\sqrt{m_c}\right)\Big|_{m_c \to 0} \sim \ln|z-z'|. \tag{19}$$

Hence the behaviour of the correlation functions (12), (18) is consistent with (14), (15) for the values $S = \{1, 2\}$.

The conformal invariance of the random walk [5,6] together with the geometrical interpretation of the monodromy properties of the four-point conformal correlator established above enable us to express the following.

Statement. The critical conformal field theory characterized by the values $c = -2$ and $\Delta = -\frac{1}{8}$ gives the second quantized representation for the random walk of the infinite length on the double-punctured complex plane.

The conformal field theory with $c = -2$ and $\Delta = -\frac{1}{8}$ has been studied recently in [13] as an example of the non-minimal model with the logarithmic-like singularities in the correlation functions. Thus the behaviour (19) could be considered as an additional confirmation of the statement expressed above.

5. Topological invariants from the conformal approach and Alexander algebraic invariants for braid B_3

Consider the braid group B_3 defined by the following relations among generators $\{\mu_1, \mu_2\}$:

$$\mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2 \tag{20a}$$

$$\mu_1\mu_1^{-1} = \mu_1^{-1}\mu_1 = \mu_2\mu_2^{-1} = \mu_2^{-1}\mu_2 = 1\,. \tag{20b}$$

It is known [14] that the relations (20a, b) are satisfied for μ_i ($i = 1, 2$) written in the so-called Magnus matrix representation

$$\mu_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix} \tag{21}$$

where t is the 'spectral parameter'.

The Alexander algebraic invariant of the braid B_3 can be constructed in the following way. Let us consider an arbitrary braid of length L; let it be, for example,

$$\mu^{(L)} = \overbrace{\mu_1\mu_2\mu_1^{-1}\mu_2^{-1}\mu_1^{-1}\ldots}^{L \text{ elements}} \tag{22}$$

where $\mu^{(L)}$ is the (2×2)-matrix obtained by multiplications of all L elementary matrices along the braid, i.e. $\mu^{(L)}$ is a particular representation of the Markov chain.

The Alexander polynomial invariant now reads [14]

$$Al\{t\} = \det\left[\mu^{(L)} - E\right]\,. \tag{23}$$

We prove in [3] the following.

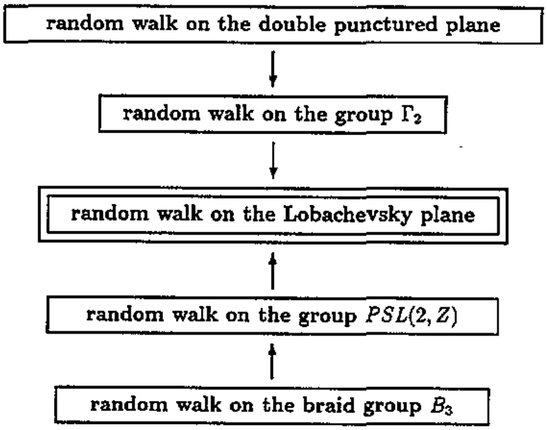

Figure 4. Relations between limiting behaviours of the random walks on Γ_2, B_3 and on the Lobachevsky plane.

Theorem.

(1) Consider the Markov chain of length L with the uniform transition probabilities equal to $\frac{1}{4}$ defined on the set of B_3 generators $\{\mu_1, \mu_2, \mu_1^{-1}, \mu_2^{-1}\}$. Define the highest power, $m_{Al}^{\max}(L)$, of the Alexander invariant

$$m_{Al}^{\max}(L) = \lim_{t \to \infty} \frac{\ln Al\{t\}}{\ln t} \tag{24}$$

which plays the role of Lyapunov exponent of the matrix product (22) [15].

(2) Consider the random walk of length L which starts at the point z and finishes at the point z' on the double-punctured plane $\mathfrak{R}$; denote this path $C(L)$. Define the non-Euclidean distance $|\eta(L)|$ between the ends of the path $C(L)$ in the covering space $\mathfrak{I}$ (the upper half-plane $\operatorname{Im}\tau > 0$),

$$|\eta(L)| = |f_{\text{in}}(z) - f_{\text{fin}}(z')| = |\operatorname{Inv}(C(L))| \tag{4$'$}$$

(compare to (4)).

Now the following relation is valid:

$$\lim_{L \to \infty} \frac{m_{Al}^{\max}(L)}{|\eta(L)|} = \text{constant}. \tag{25}$$

The proof is based on the fact that the group B_3 is the central extention of the group $PSL(2, Z)$ generated by the matrices (21) for $t = -1$ and without loss of generality we can consider the random walk just on $PSL(2, Z)$. The group $PSL(2, Z)$ has the representation in the upper half-plane $\operatorname{Im}\tau > 0$ with the fundamental domain in the form of a triangle with angles $\{0, \pi/3, \pi/3\}$ [7] (compare to the corresponding construction for the group Γ_2). From figure 4 it is easy to see that the limit behaviour of the random walks on $PSL(2, Z)$ and on Γ_2 are similar because in both cases they are governed by the Laplacian on the Lobachevsky plane. This is the origin of the relation (25). The complete proof of (25) and detailed consideration of the random walk on $PSL(2, Z)$ is given in [3]. Let us note that the highest power of the Alexander polynomial, $m_{Al}^{\max}(L)$, characterizes the 'complexity' [16] of the braid and can be used for the rough classification of the homotopy class of the braid B_3.

6. Conclusion

Instead of discussion we formulate, without proof, the theorem which generalizes the statement of section 5.

Theorem.

(1) Consider the random braid B_n of length L, i.e. define the uniform distribution on the set of braid generators $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\, \mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_{n-1}^{-1}\}$ and construct the Markov chain of length L from these generators. The topological state of the braid B_n we characterize by some algebraic polynomial invariant (Alexander, Jones, HOMFLY,...). Define the 'Lyapunov exponent'—the highest power, $m^{\max}(L)$, of these polynomials.

(2) Consider the random walk of length L on the Lobachevsky plane with the scalar curvature $\lambda = \lambda(n)$ (where n is the number of strings in the braid). The non-Euclidean distance between the ends of the random walk of length L we denote by $|\eta(L, \lambda)|$.

Now the following relation is valid:

$$\lim_{L\to\infty} \frac{m^{\max}(L)}{|\eta(L, \lambda)|} = \text{const}(\lambda)\,. \tag{26}$$

Acknowledgments

We are grateful to Vl Dotsenko, J Therry-Mieg and B Duplantier for helpful discussions of the field–theoretic aspects of the work and to Ya Sinai, A Grosberg and P Kalugin for valuable useful comments. We would like to thank the CNRS (France) for financial support in the framework of the 'Landau-Steklov' program.

References

[1] Polyakov A M 1988 *Mod. Phys. Lett.* A 3 325
[2] Witten E 1989 *Commun. Math. Phys.* **121** 351
[3] Nechaev S K, Grosberg A Yu, Vershik A M 1994 *Phys. Rev. Lett.* in press
[4] Belavin A A, Polyakov A M, Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[5] Nechaev S K 1988 *J. Phys. A: Math. Gen.* **21** 3659
[6] Ito K And McKean H P 1965 *Diffusion Processes and Their Sample Paths* (Berlin: Springer)
[7] Mumford D 1983 *Tata Lectures on Theta I, II (Progress in Mathematics 28, 34)* (Boston, MA: Birkhauser)
[8] Golubev V V 1950 *Lectures on Analytic Theory of Differential Equations* (Moscow: GITTL)
[9] Dotsenko Vl S 1984 *Nucl. Phys.* B **235** 54
[10] Knizhnik V G and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[11] Duplantier B and David F 1988 *J. Stat. Phys.* **51** 327
[12] Nechaev S K, Semenov A N, Koleva M K 1987 *Physica* **140A** 506
Koralov L B, Nechaev S K and Sinai Ya G 1993 *Prob. Theor. Appl.* **38** 331 (in Russian)
[13] Gurarie V 1993 *Preprint* PUPT-1391
[14] Birman J 1975 *Braids, Links and Mapping Class Groups (Ann. Math. Stud. No 82)* (Princeton, NJ: Princeton University Press)
[15] Nechaev S K and Sinai Ya G 1991 *Bol. Soc. Bras. Mat.* **21** 121
[16] Grosberg A Yu and Nechaev S K 1992 *Europhys. Lett.* **20** 613